Chapter 2. Electrostatics

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***Introduction to Electrodynamics*, 3rd or 4rd Edition, David J. Griffiths**

2.3 Electric Potential

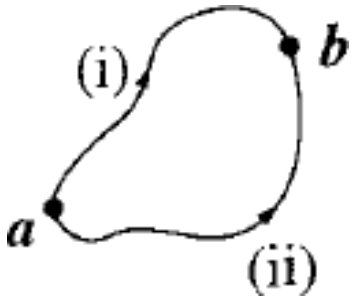
2.3.1 Introduction to Potential

We're going to reduce a vector problem (finding **E** from $\nabla \times \mathbf{E} = 0$) down to a much simpler scalar problem.

$\nabla \times \mathbf{E} = 0$ ⇨ $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ ➔ **the line integral of E from point a to point b is the same for all paths (independent of path)**

Because the line integral of **E** is independent of path, we can define a function called the **Electric Potential**:

$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$: O is some standard reference point



➔ The ***potential difference*** *between two points a and b is*

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

➔ The fundamental theorem for gradients states that $V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l}$

⇨ $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$ ⇨ $\boxed{\mathbf{E} = -\nabla V}$

➔ **The electric field is the gradient of scalar potential**

2.3.2 Comments on Potential

$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

(i) The name.

➔ "**Potential**" and "**Potential Energy**" are completely different terms and should, by all rights, have different names.

➔ There *is a* ***connection*** between "potential" and "potential energy":

$$W = QV(\mathbf{r})$$

Ex:

(ii) Advantage of the potential formulation.

$$\mathbf{E} = -\nabla V$$

➔ ***"If you know V, you can easily get E"*** by just taking the gradient:

➔ This is ***quite extraordinary***:

➔ *One can get a* ***vector quantity E (three components) from a scalar V (one component)!***

➔ Ho[illegible] ***inde***[illegible]***ions*** carr

$$\nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \qquad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

➔ The answer is that ***the three components of E are not really independent***.

➔ Therefore, ***E is a very special kind of vector***: whose curl is always zero

Comments on Potential $V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$

(iii) The reference point *O*.

➔ ***The choice of reference point 0 was arbitrary*** → "ambiguity in definition"

➔ Changing reference points amounts to adding a constant *K to the potential:*

$$V'(\mathbf{r}) = -\int_{\mathcal{O}'}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{O}'}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = K + V(\mathbf{r})$$

➔ Adding a constant to *V will not affect the potential difference:*

$V'(\mathbf{b}) - V'(\mathbf{a}) = V(\mathbf{b}) - V(\mathbf{a})$ → since the *added constants cancel out.*

➔ Nor does the ambiguity of adding constants affect the gradient of *V:*

$\nabla V' = \nabla V$ → since the *derivative of a constant is zero.*

➔That's why **all such *V's***, *differing only in their* choice of reference point, correspond to **the same field E**.

➔ Ordinarily, we "**set the zero of potential at infinity**." $V(\mathcal{O}) = 0$
Choosing a reference point at a place where *V is to be zero.*

Comments on Potential $V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$

(iv) Potential obeys the superposition principle.

➔ *Because* the electric field *obeys the superposition principle:* $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots$

➔ Integrating from the common reference point to r, the potential also satisfies

$$V = V_1 + V_2 + \ldots$$

➔ The potential is the **sum of the potentials** due to all the source charges.

➔ It is an ***ordinary sum***, ***not a vector sum***, *it makes* a lot easier to work with.

(v) Units of Potential.

➔ Newton-meters per coulomb or Joules per coulomb (J/C = **volt**)

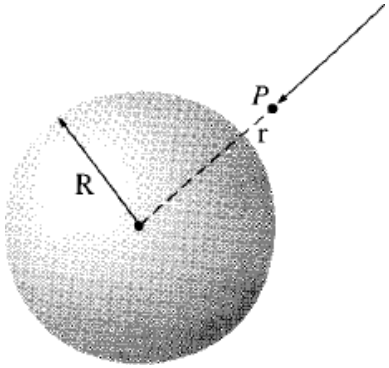
Example 2.6 Find the potential inside and outside a spherical shell of radius *R with a* uniform surface charge of total q.

From Gauss's law, the field outside is $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$

$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\Big|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

The field inside is zero:

$$V(r) = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{R} \frac{q}{r'^2}\,dr' - \int_{R}^{r} (0)\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\Big|_{\infty}^{R} + 0 = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$



2.3.3 Poisson's Equation and Laplace's Equation

➔ **What do the fundamental equations for E look like, in terms of *V*?**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \xrightarrow[\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V]{\mathbf{E} = -\nabla V} \nabla^2 V = -\frac{\rho}{\epsilon_0}$$

: Poisson's equation

$$\nabla^2 V = 0$$

: Laplace's equation

➔ **Gauss's law on E can be converted to Poisson's equation on V**

$$\nabla \times \mathbf{E} = 0 \xrightarrow[\nabla \times \mathbf{E} = \nabla \times (-\nabla V)]{\mathbf{E} = -\nabla V} \nabla \times (-\nabla V) = 0$$

➔ That's no condition on *V since the curl of gradient is always zero.*

$$\nabla \times \mathbf{E} = 0 \textit{ permits } \mathbf{E} = -\nabla V \longleftrightarrow \mathbf{E} = -\nabla V \textit{ guarantees } \nabla \times \mathbf{E} = 0$$

➔ It takes only one differential equation (Poisson's) to determine V, because V is a scalar;
(for E we needed two, the divergence and the curl.)

2.3.4 The Potential of a Localized Charge Distribution

$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

Based on the assumption that the reference point is at infinity.

The potential of a point charge *q:* $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$

The potential of a collection of point charges: $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$

The potential of a continuous charge distribution: $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r}dq$

For a volume, surface, or line charge → $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{r}d\tau'$ $\frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{r}da'$

$\frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{r}dl'$

(Note) Compare it with the corresponding E formula:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{r}d\tau' \quad \longleftrightarrow \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{r^2}\hat{\mathbf{r}}d\tau'$$

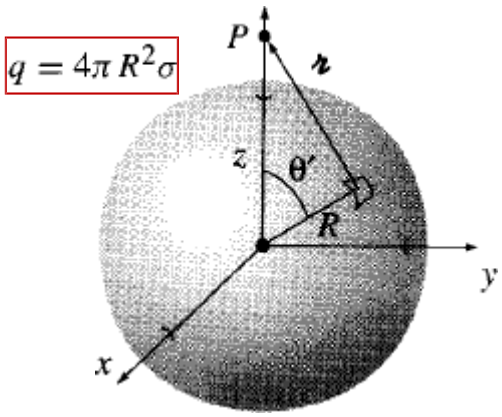
➔ **The main point to notice is that the unit vector $\hat{r}$ *is now missing,***
➔ ***so there is no need* to *worry about components.***

The Potential of a Localized Charge Distribution

Example 2.7 Find the potential of a uniformly charged spherical shell of radius *R*.

In Example 2.6, we used Gauss's law:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \longrightarrow V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$



But, now let's use the expression of charge distribution:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}}\,da' \xrightarrow[da' = R^2\sin\theta'\,d\theta'\,d\phi']{\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'} 4\pi\epsilon_0 V(z) = \sigma\int \frac{R^2\sin\theta'\,d\theta'\,d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= 2\pi R^2\sigma\int_0^{\pi}\frac{\sin\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}\,d\theta'$$

$$= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

$$V(z) = \frac{R\sigma}{2\epsilon_0 z}[(R+z)-(z-R)] = \frac{R^2\sigma}{\epsilon_0 z}, \quad \text{outside;} \xrightarrow[q = 4\pi R^2\sigma]{} V(z) = (1/4\pi\epsilon_0)(q/z)$$

$$V(z) = \frac{R\sigma}{2\epsilon_0 z}[(R+z)-(R-z)] = \frac{R\sigma}{\epsilon_0}, \quad \text{inside.}$$

5. Summary; Electrostatic Boundary Conditions

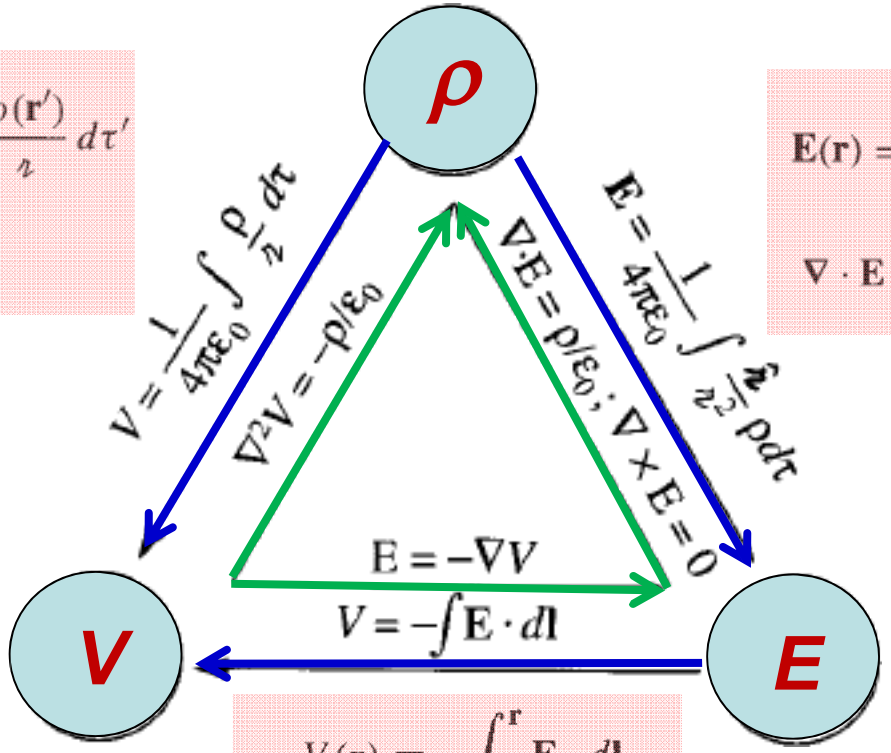
From just two experimental observations:

(1) the principle of superposition - a broad general rule

(2) Coulomb's law *- the fundamental law of electrostatics.*

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\imath}\, d\tau'$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$



$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\imath^2}\hat{\imath}\, d\tau'$$

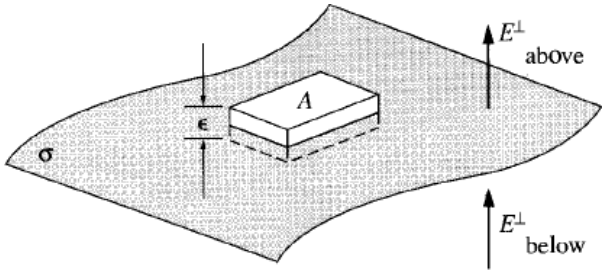
$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\times\mathbf{E} = 0$$

$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}$$

$$\mathbf{E} = -\nabla V$$

2.3.5 Summary; Electrostatic Boundary Conditions

Notice that the electric field always undergoes a discontinuity when you cross a surface charge σ.

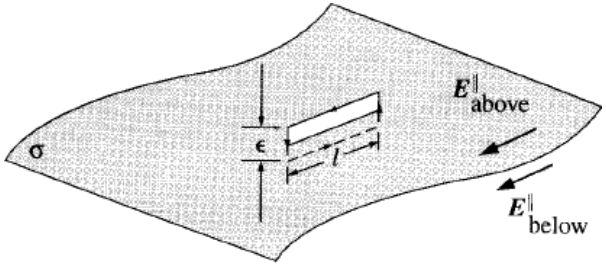


$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \sigma A$$

As the thickness ε goes to zero,

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0} \sigma$$

➔ *The normal component of **E** is discontinuous by σ / ε_0 at any boundary.*



➔The *tangential component of **E**,* by contrast, is always continuous.

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}}$$

➔ The boundary conditions on E into a single formula:

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$$

➔ The potential, meanwhile, is continuous across any boundary:

$$V_{\text{above}} - V_{\text{below}} = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \longrightarrow V_{\text{above}} = V_{\text{below}}$$

as the path